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# Does Multiplicity of Equilibria Arise in the Eaton-Gersovitz Model of Sovereign Default?

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# Does multiplicity of equilibria arise in the Eaton-Gersovitz model of sovereign default? *

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**Abstract**

No, at least for a rich parameter space. A common view within the class of sovereign default models is they are subject to multiple equilibria. This paper quantitatively analyzes such claims by using the model and the extensions of Eaton and Gersovitz (1981); a benchmark sovereign default model for quantitative investigation of endogenous default risks. It is shown that within the confines of a rich parameter space the issue of multiplicity never arises in the model simulations when the government debt has one-period or long-term maturity. This paper also shows that inclusion of renegotiation process for endogenous debt recovery to these models as well as inflation and non-defaultable debt along with non-state contingent defaultable debt do not generate multiplicity. This paper sharpens our understanding of such models and presents that the quantitative implications of the literature following these models are not byproduct of bad equilibrium selection.

**Keywords:** sovereign default, multiple equilibria
**JEL Codes:** E44, F34

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# 1 Introduction

This paper provides a quantitative analysis of multiplicity arising from sovereign default models. In particular, this paper focuses on the model of Aguiar and Gopinath (2006) and Arellano (2008) for the analysis of one-period debt and Hatchondo and Martinez (2009) for the analysis of long-term debt which all build on Eaton and Gersovitz (1981) class of models. I then extend these models by including positive recovery after a default which was the subject of Yue (2010) and study whether positive recovery assumption generates different multiplicity dynamics.

A number of studies have recently emerged which particularly study the multiplicity of equilibrium in Eaton-Gersovitz (EG) model as well as the slight variants of it. To name a few: Lorenzoni and Werning (2013), Auclert and Rognlie (2015) and Ayres et al. (2015). One common theme of the recent papers raising multiplicity view the byproduct of the policy responses of the European Central Bank as having eliminated bad equilibrium selection[1].

The objective of this paper is to sharpen our understanding of EG models for quantitative studies and investigate whether there exists a multiplicity region which would raise concerns for the implications of quantitative articles that follow EG models.

Formally, as in the classic EG model and subsequent papers this paper features a small open economy model where the agents receive stochastic stream of income and the government's objective is to maximize the utility of the representative household. Each period the government decides whether to default on its debt or to repay. Upon repayment, the government can issue new debt and if a government repudiates the debt, it faces a direct income cost and is temporarily excluded from issuing new debt and regains access for stochastic number of periods with zero debt. I then extend the baseline model by relaxing some of the assumptions to account for the subsequent developments in the literature, as

[1] The literature considers that the decline in the sovereign borrowing costs and the subsided default fears are the joint product of following forces: Mario Draghi's "whatever it takes" speech in July 2012 as well as the government bond purchase programmes such as Securities Market Programme (SMP) and the Outright Monetary Transactions (OMT). These papers consider that the channel works through shutting down the bad equilibrium territory.

described below.

In this paper, I first quantitatively show that the model simulations of one-period benchmark model do not yield any single adverse equilibrium. My findings thus confirms the theoretical findings of Auclert and Rognlie (2015), that is the equilibrium in one-period debt model is unique. I then consider a case where the debt is modeled as long-term debt for which Auclert and Rognlie (2015) fail to show neither uniqueness nor multiplicity that long-term debt models might inherit. I show that multiplicity never arises in model simulations by iterating EG operator using "good" and "bad" initial conditions. Both models yield nearly identical functional values as well as decision rules and they also generate the same debt and business cycle moments.

One of the main results of this paper is that the quantitative findings of EG literature are not the byproduct of bad equilibrium selection and multiplicity is not necessarily a feature of EG models. To establish that model assumptions are relaxed one by one and multiplicity is studied. I then enrich the model with further extensions and investigate whether such extensions that are prevalent in the literature generate any multiplicity. I initially consider a case where value of default is no longer exogenous (a typical setting of EG models) and both the timing and amount of recovery is endogenously determined through a bargaining game. I subsequently run a battery of different modalities of EG models that exist in the literature and investigate whether the introduction of inflation or one-period non-defaultable debt generate multiplicity. Furthermore, I relax the assumption of risk-neutral lenders (a common sovereign default model set up in the literature) and show that any of these extensions also do not generate multiplicity in the model simulations. Finally, this paper considers different sovereign debt models that do not originate from EG models and discusses how the timing and commitment assumptions may play out a role for generating multiplicity (see Calvo (1988) and Cole and Kehoe (2000)).

This paper contributes to our understanding of EG models because it quantitatively shows that multiplicity concerns are not valid for these benchmark sovereign default models. The results are also directly applicable to wide range of models that are extending the EG model to quantitatively study a plethora of questions. This paper therefore confirms that the findings of quantitative sovereign debt literature that extends EG models are not driven by adverse equilibrium selection.

This paper does not rule out multiplicity in sovereign default models and does not reject that the policy responses of the ECB eliminated the bad equilibrium. It might be possible to generate multiplicity for a different set up with different assumptions. Rather, this paper contributes to our understanding of EG models and paves the way for rethinking the necessary assumptions to generate multiplicity.

The layout of this paper proceeds as follows; Section 2 presents the benchmark model, Section 3 introduces the calibration, Section 4 presents the simulation results and Section 5 features renegotiation stage and endogenous recovery of the debt. Section 6 outlines more extensions of EG model and Section 7 concludes.

# 2 Model

This section lays out a dynamic small open economy model in which the government issues non-state-contingent defaultable long-term debt. One period debt would be a special case of the model's specification.

## 2.1 Model timeline

The timing of events is summarized as follows:

1. Period $t$ starts and the government's endowment $y$ and indebtedness $b$ are known by each agent in the economy.
2. The government first chooses whether to default or not:
    - Upon repayment, the government chooses how much debt to hold for the next period.

- If the government does not honor its debt obligations, it is excluded from the credit markets for a constant period of time and barred from issuing debt. The government is subject to an income cost of defaulting during its exclusion and regains access with zero debt, so there is no recovery in the benchmark model. These assumptions will be relaxed later in the paper.

3. Period $t+1$ starts.

## 2.2 Environment

**Preferences and endowment.** Benevolent government's objective is to maximize the utility of the representative household:

$$E_0 \sum_{t=0}^{\infty} \beta^t u(c_t) \tag{2.1}$$

where $E$ denotes the expectation operator, $0 < \beta < 1$ denotes discount factor and $c_t$ denotes aggregate consumption at time $t$. The utility function $u(.)$ belongs to the class of CRRA utility functions and read as:

$$u(c) = \frac{c^{1-\gamma} - 1}{1-\gamma}. \tag{2.2}$$

Utility function $u(.) : [0, \infty) \to \mathcal{R}$ is increasing, strictly concave, continuous, and bounded above by the quantity $U$ and $\gamma$ denotes the risk aversion parameter.

The households receive an income of the consumption good $y \in \mathcal{Y} \subset \mathbb{R}_{++}$ which follows a stochastic Markov process:

$$\log(y_t) = (1-\rho)\,\mu + \rho \log(y_{t-1}) + \varepsilon_t,$$

with $|\rho| < 1$, and $\varepsilon_t \sim N\left(0, \sigma_\epsilon^2\right)$.

**Asset space.** As in Hatchondo and Martinez (2009), Chatterjee and Eyigungor (2012) and Arellano and Ramanarayanan (2012), I assume that a defaultable bond issued in period

$t$ promises an infinite stream of coupons payments $\kappa$ which decreases at a constant rate $\delta$ with a final price of zero $(\kappa, (1-\delta)\kappa, (1-\delta)^2\kappa, ..., 0)$. In particular, a defaultable bond issued in period $t$ promises to pay $\kappa(1-\delta)^{j-1}$ units of the tradable good in period $t+j$, for all $j \geq 1$ where $\delta \in (0,1]$ and $\kappa = \frac{r+\delta}{1+r}$ are fixed parameters. Hence, long-term debt dynamics can be represented as follows:

$$b_{t+1} = (1-\delta)b_t + i_t,$$

where $b_t$ is the number of defaultable coupon claims at the beginning of period $t$, and $i_t$ is the number of defaultable long-term bonds issued in period $t$. Notice that $\delta = 1$ corresponds to the one-period debt of the economy. The advantage of modeling long-term debt this way enables to keep track of the model dynamics and avoids including large number of state variables for different bond maturities.

The budget constraint of the economy conditional on having an access to credit markets is given by:

$$c_t = y_t - \kappa b_t + q_t i_t,$$

where $q_t$ is the price of the bonds issued by the government and will depend on the borrowing rule for the long-term debt as well as the income shocks in equilibrium.

**Defaults.** When a government defaults, it is a common assumption in the literature and also contractually the case that defaulting government's all future debt obligations becomes due. As in most of the studies in the sovereign default literature, it is assumed that lenders cannot recover any defaulted debt, in other words, the fraction of the defaulted debt the lenders can recover is zero. Zero recovery assumption is relaxed in Section 5 and the multiplicity implications with positive recovery is provided.

It is also a common assumption in most of the sovereign default papers that a default event leads an exclusion from credit markets for stochastic number of periods and the government gains access to the markets with an exogenous probability of $\psi \in [0,1]$. This assumption is relaxed later in the model and a model with endogenous recovery is provided. The government also suffers an income loss of $\phi^d$ for every period during an exclusion. Thus, the budget constraint of an economy in a default becomes:

$$c_t = y_t - \phi^d(y).$$

**Equilibrium concept.** The government lacks commitment technology, that is, the government cannot commit to its strategy on debt repayment and borrowing. This paper focuses on Markov Perfect Equilibrium (MPE). Hence, equilibrium decisions depend only on the current payoff-relevant state variables.

## 2.3 Recursive formulation

Let $v$ denote the value function of a government that has the default option, $v_R(b, y)$ be the value of repayment and $v_D(y)$ be the value of defaulting. For any price function $q$, the function $v$ satisfies the following functional equation:

$$v(b, y) = Max \{v_R(b, y), v_D(y)\}, \tag{2.3}$$

where the government's value of repaying is given by

$$\begin{aligned} & v_R(b, y) = \max_{b' \geq 0} \left\{ u(c) + \beta \mathbb{E}_{y'|y} v(b', y') \right\}, \\ & \text{subject to} \\ & c = y - \kappa b + q(b', y) \left[ b' - (1-\delta) b \right]. \end{aligned} \tag{2.4}$$

Since the government does not make any payments to private creditors and regains access to credit markets with zero recovery rate, the value of default $v_D$ does not depend on the long-term debt level. Thus, the value of defaulting is given by:

$$\begin{aligned} & v_D(y) = u(c) + \beta \mathbb{E}_{y'|y} \left[ (1-\psi) v_D(y') + \psi v(0, y') \right], \\ & \text{subject to} \\ & c = y - \phi^d(y). \end{aligned} \tag{2.5}$$

The solution to the government's problem yields a default decision rule $\hat{d}(b,y) \in \{0,1\}$ (1 for default, 0 for repayment) and borrowing rule for long-term debt $\hat{b}$. Lenders are assumed to be risk neutral and sovereign debt market is competitive. Using these decision rules and the assumption on lenders, the price is determined in equilibrium (defined in Section 2.4). The price of the bond $q$ equals to the discounted expected value of all future cash flows:

$$
\begin{aligned}
q(b',y) &= \mathbb{E}_{y'|y}\left[\frac{\left(1-\hat{d}\left(b',y'\right)\right)\kappa}{1+r} + \frac{\left(1-\hat{d}\left(b',y'\right)\right)\left(1-\hat{d}\left(b'',y''\right)\right)\kappa(1-\delta)}{(1+r)^2}, \ldots\right] \\
&= \frac{\mathbb{E}_{y'|y}\left[\left(1-\hat{d}\left(b',y'\right)\right)\left[\kappa+(1-\delta)q\left(b'',y'\right)\right]\right]}{1+r}, \qquad (2.6)
\end{aligned}
$$

where

$$b'' = \hat{b}(b',y').$$

Equation (2.6) indicates that in equilibrium, the value of selling a bond today and investing it in a risk-free asset (left-hand side of the equation) has to be equal to the expected value of keeping the bond (right-hand side of the equation). If the lender keeps the bond and the government does not default next period (i.e. $\hat{d}(b',y') = 0$), he first receives one unit coupon payment $\kappa$ and he can sell the remaining receivables at market price which is equal to $(1-\delta)$ times the price of a bond issued next period $q\left(b'',y'\right)$.

## 2.4 Definition of Equilibrium

This paper focuses on Markov Perfect Equilibrium (MPE) wherein the government's equilibrium default and borrowing decisions are functions of payoff relevant state variables.

**Definition 1.** *A Markov Perfect Equilibrium is characterized by a collection of value functions $v$, $v_R$ and $v_D$; default rule $\hat{d}$, and future borrowing rule $\hat{b}$; and a bond price function $q$ such that:*

*i. given a price function $q$; the value functions $\{v, v_R, v_D\}$ and the policy functions $\left\{\hat{d}, \hat{b}\right\}$ solve the Bellman equations (2.3), (2.4), and (2.5).*

*ii. given policy rules* $\left\{\hat{d}, \hat{b}\right\}$*, the price function q satisfies condition (2.6).*

# 3 Parametrization

Following Arellano (2008), this paper assumes the cost of defaulting increases more than proportionally with income and similar to Chatterjee and Eyigungor (2012), I assume a quadratic loss function for income during a default episode $\phi^d(y) = d_0 y + d_1 y^2$. Quadratic loss function ensures that the default likelihood during high income states is lower.

The model is calibrated for Mexico and its stochastic process for GDP is estimated from the periods of 1980Q1-2007Q1. The estimated parameters from the stochastic process are similar if different countries are used such as Argentina which is also subject of many quantitative sovereign default models for calibration. Following a number of studies in the literature the probability of regaining access parameter $\psi$ is set to be 0.282. For debt duration $\delta$ is calibrated to be 0.0341 which delivers an average duration of 4 years in the simulations. It is consistent with emerging market economies' sovereign debt duration documented by Cruces et al. (2002). Eventually in robustness analysis a number of different parameters will be employed to check for multiplicity dynamics. As standard in the long-term sovereign debt literature (see Hatchondo and Martinez (2009)), Macaulay definition of duration is used in this paper and it is given by $D = \frac{1+r^*}{\delta+r^*}$ where $r^*$ denotes the periodic yield delivered by the bond.

In the literature, the parameters for the cost of income and the discount factor $\beta$ are used to match the equilibrium mean levels of sovereign debt and spreads. The parameters $d_0$ and $d_1$ are the direct costs of defaulting on income and I set $d_0$ and $d_1$ to be -0.69 and 1.027, respectively. Table 1 summarizes the parameters used in the benchmark analysis. In the simulations I obtain a mean debt-to-income ratio of 41.4 percent and a mean spread of 5.3 percent [2].

[2] To compute the sovereign spreads in the bond price, I first compute the yield on a bond in the simulations and it is obtained as the discounted value of coupons given that the government does not default and the

Table 1: Parameter Values

| | Parameter | Value |
|---|---|---|
| Risk aversion | $\gamma$ | 2 |
| Risk-free rate | $r$ | 1% |
| Probability of reentry after default | $\psi$ | 0.282 |
| Income autocorrelation coefficient | $\rho$ | 0.85 |
| Standard deviation of innovations | $\sigma_\epsilon$ | 2.7% |
| Mean log income | $\mu$ | $(-1/2)\sigma_\epsilon^2$ |
| Debt duration | $\delta$ | 0.0341 |
| Calibrated | | |
| Discount factor | $\beta$ | 0.961 |
| Income cost of defaulting | $d_0$ | -0.69 |
| Income cost of defaulting | $d_1$ | 1.027 |

## 3.1 Computation Algorithm

The numerical algorithm to solve the model works through iterating on value functions $v_R$ and $v_D$ and price function $q$ until convergence is obtained. The algorithm works as in Hatchondo et al. (2010): It first solves for the equilibrium of the finite-horizon version of the economy, that is, the interpolated value and price functions of the first and second period are sufficiently close (smaller than $10^{-5}$) after the iterations. They then use the first-period equilibrium functions as the infinite-horizon-economy equilibrium functions.

This paper uses 21 grid points for long-term debt holdings and income realizations, and 50 Gauss-Legendre quadrature points to calculate the expectations.

coupons are held until maturity (infinity in this case). That is, given price $q$, the yield $r^*$ satisfies

$$q = \sum_{n=1}^{\infty} \frac{(1-\delta)^{n-1}}{(1+r^*)^n}.$$

So,

$$r^* = \frac{\kappa}{q} - \delta.$$

The sovereign spread is defined as the difference between the yield $r^*$ on long-term bond and the default free rate $r$ and it can be calculated as

$$r_s = \left(\frac{1+r^*}{1+r}\right)^4 - 1$$

where $r_s$ is the annualized spread reported in the tables. Debt levels reported in the simulations are calculated as the present value of future debt obligations discounted at the default free rate $r$, in particular it is calculated as $\frac{b'}{\delta+r}$.

An approximation scheme is required for the evaluation of the equilibrium value functions $v_R$ and $v_D$ and price function $q$ outside the grids and this paper engineers a spline interpolation over asset space ($b$) and linear interpolation over endowment levels. In particular, routines DCSDEC and DCSVAL are employed using the IMSL library for Fortran[3].

The numerical algorithm that is used to solve the model works as follows:

1. Guesses of $v_R$, $v_D$ and $q$ are provided as the last-period of the finite-horizon economy:
    - $v_R(b, y) = u(y - b\kappa)$,
    - $v_D(b, y) = u(y - \phi^d(y))$,
    - and $q = 0$.
2. Optimization problem defined in equation (2.4) is solved for each grid points. First, I search over 40 points for $b'$ in order to reach the maximum point of the objective function. So for a fixed $b$, I now have the optimal $b'$ which was attained by using the one-dimensional DUVMIF routine of the IMSL library for Fortran.
3. Iterate the procedure defined in 2 for equations (2.4) to (2.6) until criteria for convergence is attained.

To simulate the model I use the converged optimal decision rules, specifically do the following steps:

- Set the number of samples $N$ and the number of periods $T$ in which $T_0$ will be dropped. In particular set $N = 300$, $T = 1501$ and $T_0 = 1000$.
- Choose $y_0$ to be mean $y$, $b_0$ to be zero and draw sequences of $y_t$ for $t = 1, 2, ..., T$, using a random number generator. It is suggested to keep the draws so that the same draws can be used again for each $n \in N$. It is also assumed that there is no default at time zero.
- Trim the first $T_0$ periods to remove the influence of arbitrarily chosen initial states.

[3]DCSDEC and DCSVAL are double precision names of CSDEC and CSVAL, respectively.

The moments presented in the paper calculated from 300 samples such that each sample includes 100 periods (25 years) with no defaults and the sample period begins at least 5 years following a default so that the impact of a default event is singled out. Moments are reported using the detrended series and trends are computed using HP filter with a smoothing parameter of 1600.

# 4 Simulation Results

This section searches for multiple fixed points and checks whether all alternatives that are considered here yield the same equilibrium functions and decision rules. To start with, let's guess that for all the states of debt and income value of repayment is greater than default and thus the price of the bond becomes risk free for all states, that is $v_R(b, y) > v_D(b, y)$ and $q(b', y) = \frac{1}{r}$ for all $b$ and $y$. Let's call the equilibrium outcome of this operator as "good" equilibrium. And for the next alternative, it is assumed that value of defaulting is greater in all states and thus the price of asset is zero, that is, $v_R(b, y) < v_D(b, y)$ and $q(b', y) = 0$ for all $b$ and $y$. Outcome of this operator will be referred as "bad" equilibrium. For different modalities of the paper that will be discussed below, I will check whether above iterations always converge to the same decision rules and functional values. "Good" equilibrium differs from the "adverse" or "bad" equilibrium in the sense that it offers higher or at least the same debt prices $q$ uniformly.

## 4.1 One-period debt

This section discusses the validity of multiplicity for one-period debt and recall that $\delta = 1$ corresponds to one-period debt of the framework presented in Section 2. My results confirm the theoretical findings of Auclert and Rognlie (2015): there is a unique-equilibrium in EG type of models.

## 4.2 Long-period debt

We now turn our attention to the long-term debt where Auclert and Rognlie (2015) fail to show uniqueness. Lorenzoni and Werning (2013)'s argument implies that multiplicity is likely to be a feature of sovereign default models. Although their analysis is not one-to-one application of EG models, they point out that there exist multiple equilibrium schedules. To this end, it is important to investigate whether long-term debt models inherit multiplicity and whether the implications of such papers are driven by the presence of multiplicity.

Understanding the dynamics is crucial. Market value of long-term debt has two crucial components: $q(b',y)b'$ and $q(b',y)(1-\delta)b$. Lorenzoni and Werning (2013) call the first term "issuance Laffer curve" and the second term "stock Laffer curve". The joint dynamics of these two different terms may yield multiplicity because of the feedback mechanism coming from the stock of the debt. And this is also the part where the proof strategy of Auclert and Rognlie (2015) fails to show uniqueness for long-term debt. As shown in equation (2.6), the price of long-term debt depends on all future default decisions as well as the borrowing amounts which may carry out a feedback mechanism and generate multiplicity.

Figure 1 shows the differences in equilibrium price and repayment functions as well as the differences in decision rules all of which are obtained from the converged values of iteration procedures that are explained above. As the figure presents two iteration procedures yield almost identical equilibrium price and repayment functions. The lower panel shows the differences in decision rules. Default decision rule across all debt and income states are identical and borrowing rules are very similar, the difference is smaller than $2 \times 10^{-6}$ for all states[4].

Table 2 presents that both good and bad operator also yield the same long-run sovereign debt and business-cycle moments. The second (third) column shows the moments obtained with good (bad) equilibrium. The last column corresponds to the long-run debt and

[4]Convergence criteria is $10^{-5}$.

business-cycle moments of Mexico.

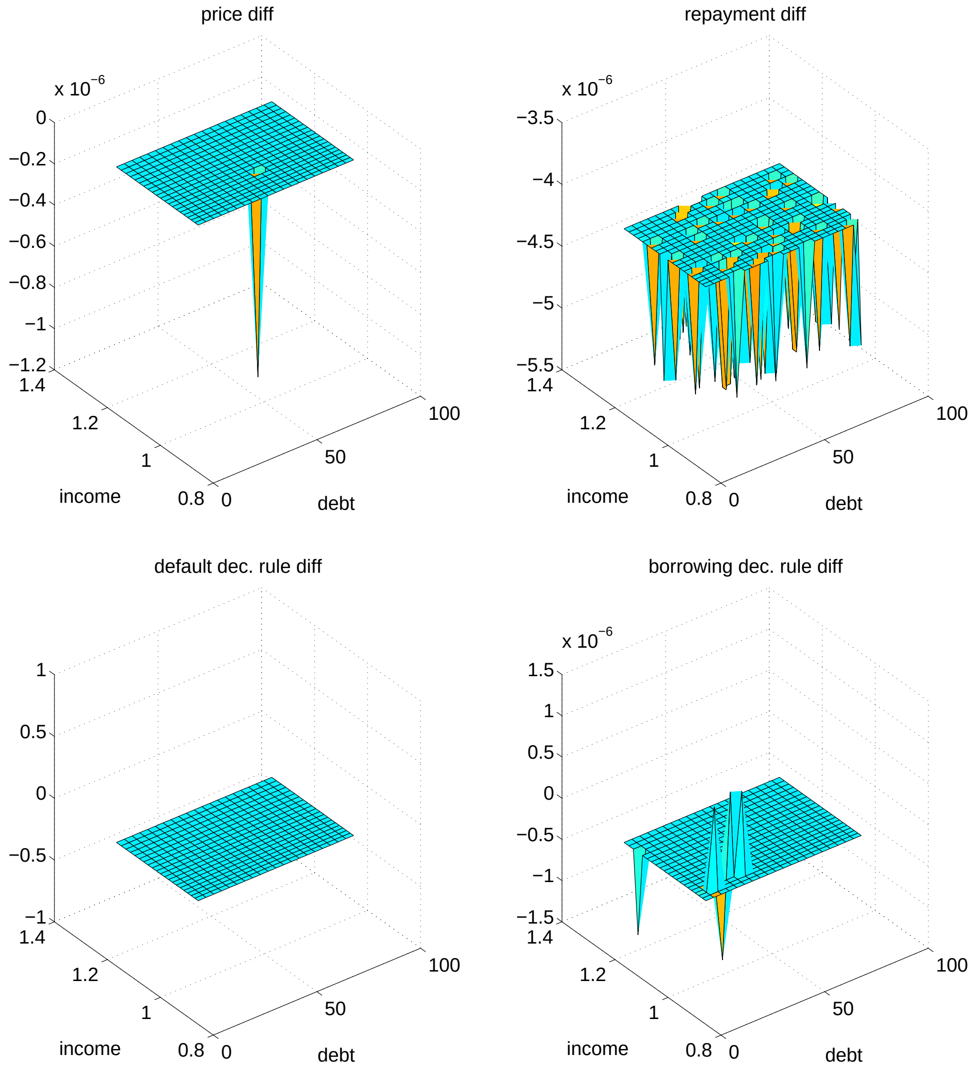


Figure 1: The upper panels show the numerical differences in price and repayment functions obtained by the two iteration procedures. The lower panels present the differences in default and borrowing decision rules obtained by the iteration procedures.

### 4.3 Robustness analysis

This section engineers a wide set of tests to check whether different sets of parameters would generate different multiplicity dynamics. I utilized wide set of parameters that govern the stochastic income process including different values of $\rho \in [0, 1)$ and $\sigma_\epsilon \in (0, 0.1)$. Also, one of the requirements of generating multiplicity in Cole and Kehoe

Table 2: Long-Run Statistics: Debt Statistics

| | Good eqm. | Bad eqm. | Data |
|---|---|---|---|
| Debt Statistics | | | |
| Mean Debt-to-GDP | 41.4 | 41.4 | 43 |
| Mean $r_s$ | 5.3 | 5.3 | 5.1 |
| $\sigma(r_s)$ | 1.67 | 1.67 | 1 |
| Default rate | 3.4 | 3.4 | 3.4 |
| Business Cycle Statistics | | | |
| $\sigma(c)/\sigma(y)$ | 1.1 | 1.1 | 1.1 |
| $\sigma(tb)$ | 0.7 | 0.7 | 1.4 |
| $\rho(c,y)$ | 0.93 | 0.93 | 0.9 |
| $\rho(r_s,tb)$ | 0.8 | 0.8 | 0.6 |
| $\rho(r_s,y)$ | -0.7 | -0.7 | -0.5 |

Standard deviation of a variable $h$ is denoted by $\sigma(h)$ and the coefficient correlation between variables $h$ and $m$ is denoted by $\rho(h,m)$. Consumption and income are reported by natural logs.

(2000) is to have high enough risk aversion and Ayres et al. (2015) use 6 for the risk aversion parameter $\gamma$ so that they have a strong preference for consumption smoothing. In robustness exercises here, $\gamma$ was also set to be 2, 4, 6 and 8. Remaining parameters to check multiplicity dynamics are $\delta$ which is used to calibrate the duration of the debt, discount factor $\beta$ and income cost of defaulting parameters $d_0$ and $d_1$. A wide set of parameters are used and different combinations of parameters are engineered; none of the specifications generate multiplicity. All different operators yield identical results.

# 5 Recovery

This section now presents a sovereign default model where the decisions of default and debt renegotiation are endogenously determined. It is shown that multiplicity is also not valid when such an extension is considered.

The value of repayment stays as in equation (2.4) and the value of defaulting can be rewritten as:

$$v_D(b,y) = u(y - \phi(y)) + \beta \mathbb{E}_{y'|y}\left[W\left(b'^D, y'\right)\right], \tag{5.1}$$

subject to

$$b'^D = b(1+r).$$

The value $W\left(b'^D, y'\right)$ is the value function of the government that does not have access to credit markets and is defined below. In a financial autarky, the government bargains with the lenders to determine a fraction $\alpha(b,y)$ of the debt to be repaid. The government can only access to markets if it reaches a deal with the lenders. The recovery rate $\alpha(b,y)$ will be determined endogeously in a Nash bargaining game to be explained in Section 5.1. The government that is in default solves the following problem:

$$W(b,y) = \max\left\{W_r(b,y), W_{nr}(b,y)\right\}, \tag{5.2}$$

where $W_r(b,y)$ and $W_{nr}(b,y)$ denote the repayment and no-repayment value functions respectively while government is in default and defined as follows:

$$\begin{aligned}
W_r(b,y) &= u(y - \alpha(b,y)b\kappa) + \beta \mathbb{E}_{y'|y} v(\hat{b}^A(b'), y'), && (5.3)\\
W_{nr}(b,y) &= u(y - \phi(y)) + \beta \mathbb{E}_{y'|y} W(b'^D, y'), && (5.4)\\
b'^D &= b(1+r), \\
\hat{b}^A(b') &= \alpha(b,y)b(1-\delta),
\end{aligned}$$

where $\hat{b}^A(b')$ denotes the post-renegotiation debt level. The price of a bond satisfies

$$q(b',y) = \frac{1}{1+r}\mathbb{E}_{y'|y}\left[\hat{d}(b',y')q^D\left(b''^D, y'\right) + \left(1 - \hat{d}(b',y')\right)\left[\kappa + (1-\delta)q\left(b'', y'\right)\right]\right], \tag{5.5}$$

where $q^D$ denotes the value of a bond in default, $b''^D = b'(1+r)$ denotes the next-period end-of-period total debt while in default, $b'' = \hat{b}'(b', y')$ denotes the next-period end-of-period debt obligations chosen by a government that repays. The function $\hat{b}'$ denotes the borrowing rule followed by governments that repay.

The government in default agrees to swap $\alpha(b, y)$ fraction of its defaulted debt for new debt. The value of a bond in a renegotiation stage satisfies

$$q^D(b', y) = \frac{1}{1+r}\mathbb{E}_{y'|y}\left[\begin{array}{l}\left(1-\hat{h}(b',y')\right)(1+r)\,q^D\left(b''^{DN}, y'\right) + \\ \hat{h}(b',y')\hat{\alpha}\left(b',y'\right)\left[\kappa + (1-\delta)q\left(b''^{DR}, y'\right)\right]\end{array}\right],$$

where $\hat{h}\left(b', y'\right)$ denotes a renegotiation decision rule and takes value 1 if a deal is reached and 0 otherwise, $b''^{DN} = b'(1+r)$ denotes the next-period end-of-period total debt when a deal is not reached, and $b''^{DR} = \hat{b}'(\hat{b}^A(b'), y')$ denotes the next-period end-of-period debt obligations chosen by a government that repays after exiting the default.

With recovery and long-term debt a notorious problem arises; consumption hikes before a default. This does not occur with one-period debt maturity, ($\delta = 1$). Government issues infinite amount of debt right before defaulting which leads to jumps in the consumption and thus problems in the value function. As a solution to this problem similar to Hatchondo et al. (2014), it is assumed that governments are not allowed to issue bonds with a price lower than $\underline{q}$ which never binds in the simulations. It is assumed $\underline{q}$ to be 0.5 so the annualized spread cannot be higher than 18.3 percent.

## 5.1 Nash Bargaining Game

Upon an agreement, current outstanding coupon claims reduced to a fraction $\theta$ of the debt $b$. Value of $W_r$ for a size of $\theta$ becomes

$$W_r(b, y) = u\left(y - \theta b\kappa\right) + \beta\mathbb{E}_{y'|y}v(\hat{b}^A(b'), y'),$$

that is the expected lifetime utility of the government that agrees to repay $\theta$ fraction of its debt. Government threats the lenders to stay in financial autarky forever and its value function in autarky is provided as:

$$v^{aut}(y) = u\left(y - \phi\left(y\right)\right) + \beta \mathbb{E}_{y'|y} v^{aut}(y').$$

So the surplus to the government from an agreement can be represented as follows:

$$\Delta^G(\theta; b, y) = W_r(b, y; \theta) - v^{aut}\left(y\right), \tag{5.6}$$

and the surplus of the agreement to the lenders is

$$\Delta^L(\theta; b, y) = \theta b \left(\kappa + (1-\delta)\, q\left(b'^{DR}, y\right)\right). \tag{5.7}$$

It is assumed that government has a bargaining power $\omega \in [0,1]$. When $\omega$ is 0, it means that lenders have all the bargaining and may recoup all of its losses. The optimal recovery rate $\alpha(b, y)$ satisfies the following equation:

$$\begin{aligned} & \alpha(b, y) \equiv \underset{\theta \in [0,1]}{\arg\max} \left[\Delta^G(\theta; b, y)\right]^{\omega} \left[\Delta^L(\theta; b, y)\right]^{1-\omega}, \\ s.t. \quad & \Delta^G(\theta; b, y) \geq 0, \\ & \Delta^L(\theta; b, y) \geq 0. \end{aligned} \tag{5.8}$$

## 5.2 Multiplicity Dynamics

Similar to Section 4, this section starts out the iteration with different initial conditions to explore the multiplicity dynamics with endogenous recovery rates. In particular, for the "good" equilibrium case, it is assumed that value of repayment is greater than value of default, and value of repayment during default is greater than not repayment for all states of the world. That is $v_R(b, y) > v_D(b, y)$, $W_r(b, y) > W_{nr}(b, y)$ and $W_r(b, y) > v^{aut}(y)$ thus, $q(b', y) = \frac{1}{1+r}$ and $\alpha(b, y) = 1 \; \forall \; (b, y)$. To compare the equilibrium outcome of this iteration, I also start with the following initial conditions such that value of repayment is smaller than value of default, and value of repayment during default is smaller than not repayment for all states of the world. That is $v_D(b,y) > v_R(b,y)$, $W_{nr}(b,y) > W_r(b,y)$ and $v^{aut}(y) > W_r(b,y)$ thus, $q(b',y) = 0$ and $\alpha(b,y) = 0 \ \forall \ (b,y)$. I call the outcome of this operation "bad" equilibrium.

Figure 2 presents that equilibrium outcomes of above iterations are almost the same for price function $q$, value function $v_R$ and repayment rate function $\alpha$. The default decision rule is identical and borrowing rule is almost identical. This exercise shows that the results obtained here are not driven by multiplicity. Also the results are robust to different specifications of bargaining power parameter $\theta$.

# 6 Further extensions of the model and multiplicity

There are a number of studies extending the work of EG models to study policy questions. This section provides examples from the literature and investigates whether these extensions generate multiplicity. I then discuss the assumptions that the literature relaxes to generate multiplicity and deviates from EG models.

**Inflation.** One strand of the literature introduces inflation to EG models and study both the inflation and default dynamics such as Fried (2012) and Hur et al. (2014). Onder and Sunel (2016) analyzes the role of inflation on default and the interactions of both. In their model, long-term debt is modeled as nominal debt rather than real debt and the governments have the option of inflating away their debt burdens. It is shown that multiplicity never arises in model simulations. For the sake of multiplicity, this model can be applicable to Hur et al. (2014) because in their specification inflation is modeled as a shock which follows the same AR(1) process with income shock while the correlation of both and income shocks are zero. So an introduction of a shock that has the same process with income would not yield different multiplicity dynamics.

**Non-defaultable debt.** Hatchondo et al. (2014) introduce non-defaultable debt along with defaultable non-state contingent debt to the EG model. They study a policy question about Eurobonds: introduction of guarantees so that risk of default would fade away and

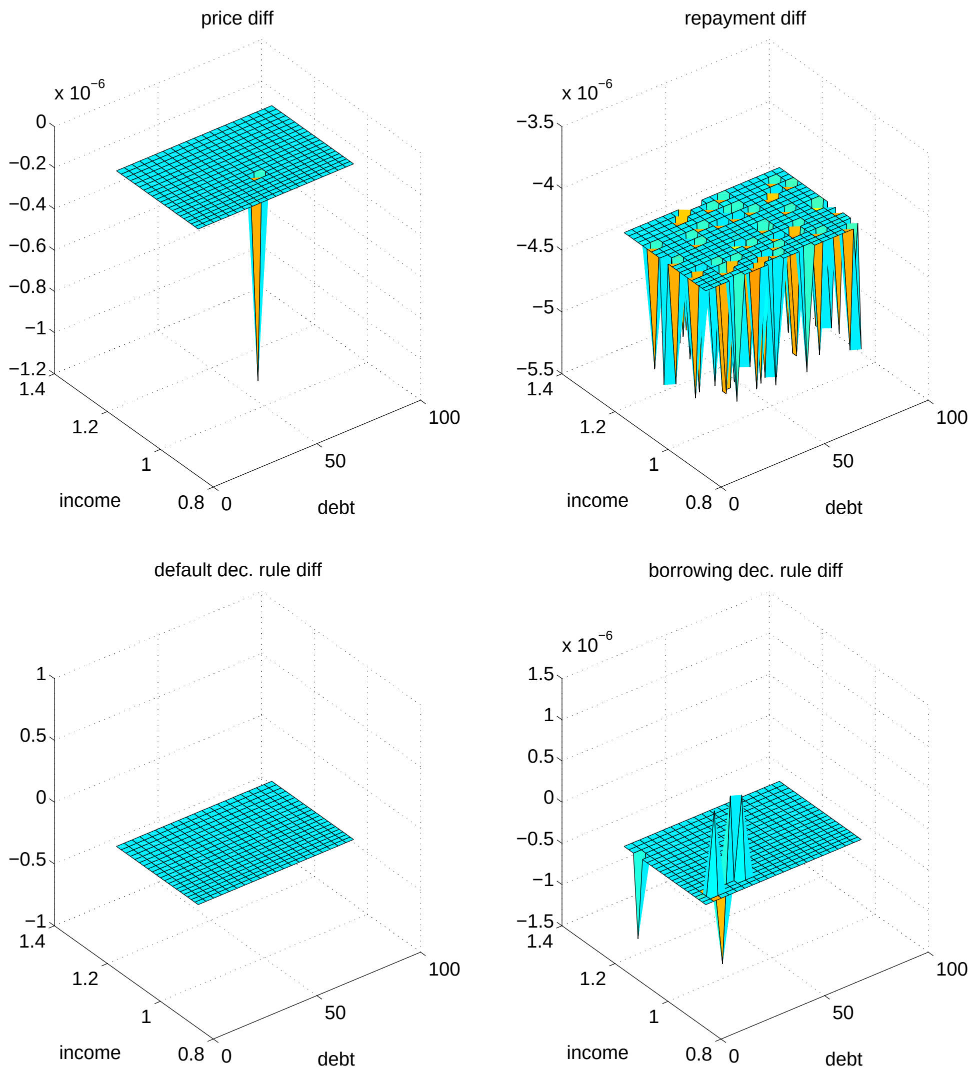


Figure 2: The upper panels show the numerical differences in price $q$, objective function $v_R$ and repayment rate function $\alpha$ obtained by the two iteration procedures. The lower panels present the differences in default and borrowing decision rules obtained by the iteration procedures.

eventually will be eliminated. They analyze the effects of such bond issuance. Onder (2015) studies the effects of arranging liquidity or swap lines when a country is hit by a sudden stop shock and shows that borrowings costs decline and governments have substantial welfare gains. The gains are significantly larger if liquidity lines are followed with fiscal rules. I show that such type of extensions do not generate multiplicity.

**Risk averse lenders.** The literature deviates from lenders' risk neutrality assumption to account for the higher sovereign risk premium demanded by the lenders. The price of the bond is determined by the no arbitrage condition with stochastic discount factor $\mathcal{G}(y', y) = exp(-r - \alpha\varepsilon' - 0.5\alpha^2\sigma_\epsilon^2)$, where $\alpha$ denotes the risk aversion parameter for lenders. In the exercises, any plausible values of $\alpha$ yields the same multiplicity intuition ($\alpha \in [0, 100]$). The above specification states that the lenders ask for a higher sovereign risk premium depending on the income shock. This formulation has advantages for replicating the spreads observed in the data and it is documented that this additional risk premium offered by the above specification accounts for the significant spread volatility in the data (see Borri and Verdelhan (2009)). With this specification the price of a bond becomes:

$$q(b', y) \quad = \quad \mathbb{E}_{y'|y}\left[\mathcal{G}(y', y)\left(1 - \hat{d}\left(b', y'\right)\right)\left[\kappa + (1-\delta)q\left(b'', y'\right)\right]\right].$$

I show that model simulations do not generate any multiplicity with this specification as well.

**Bounded savings.** Auclert and Rognlie (2015) theoretically explore whether the assumption of debt being bounded from below is important for multiplicity generation and they prove that the assumption has no bite for one-period debt. This paper also explores the importance of that assumption and let the sovereign to have an opportunitiy to save and make gains at a risk-free rate. In particular, for $b < 0$ the price of the asset becomes $\frac{1}{1+r}$. This paper quantitatively confirms the results of Auclert and Rognlie (2015) and additionally finds that the relaxation of the assumption does not lead to multiplicity for long-term debt as well. Also, this consideration does not lead to any different long-run business cycle and debt moments; it generates the same results as in Table 2.

**Multiplicity.** The role of timing and commitment assumptions may play out a role for generating multiplicity. In the model of Calvo (1988), Cole and Kehoe (2000) and more recently Lorenzoni and Werning (2013) and Ayres et al. (2015) the rules of the game are modified than that of Eaton and Gersovitz (1981). In EG models, countries default upon observing the income realization, however in Cole and Kehoe (2000) set up the government chooses to default after observing the auction outcome. If the revenue function which is computed as the price times the amount of debt issuanc from a bond-price auction is not high enough, the government keeps the proceeds of the auction and does not honor its obligations at maturity. The multiplicity or self-fulfilling crises arises following the coordination failure among lenders: with high enough stock of debt holdings $b$, lenders may not be willing to rollover existing debt by charging very high interest rates which induces and justifies default.

In its two-period game, the Calvo model borrows an exogenous amount of $b$ at time 0 and promises to pay $\mathcal{R}b$ at time 1. The government then uses distortionary taxation together with debt repudiation in order to fund its expenses. The government may partially default on its debt at time 1 and inverted U shape (calls it laffer curve due to its shape) arises stemming from the fact that lenders require higher returns when the amount of debt issuance increases.

Lorenzoni and Werning (2013) renew their focus on the necessary conditions for generating multiplicity. The fundamental difference comes with the timing of the game. Their model does not follow an optimization routine; instead they follow a fiscal rule and the focus is on the coordination issues that the lenders inherit. They show that within the confines of such changes in the game, multiplicity arises.

# 7 Conclusion

While there is a widespread perception that multipicity is a feature of sovereign default models, this paper quantitatively shows that the EG model and several extensions of it do not carry multiplicity. This study establishes that the quantitative implications of such papers are not driven by adverse equilibrium and calls for regenerated ideas for sovereign default models to induce multiplicity.

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